EE313 Linear Systems and Signals Fall 2001

Review Problems for Midterm #2 Prof. Brian Evans

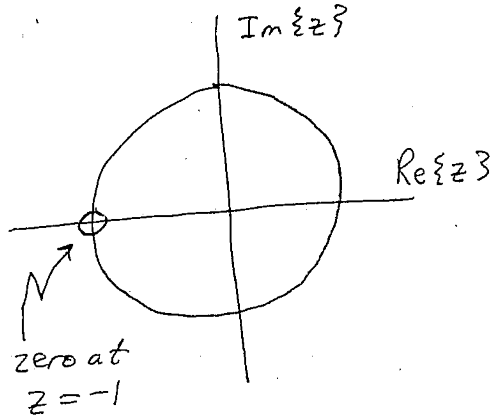
1. More explanation of midterm #2, problem #3(c)i.

The transfer function is $H[z] = \sum_{m=0}^{N-1} a_m z^{-m}$

For $N=2$, $H[z] = a_0 + a_1 z^{-1}$.

i. $a_0 = 1$, $a_1 = 1$. $H[z] = 1 + z^{-1}$

Zero occurs at $1 + z^{-1} = 0 \Rightarrow z = -1$



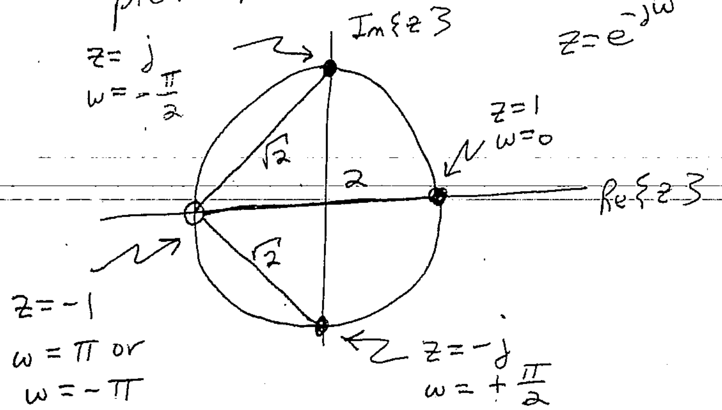
The frequency response is

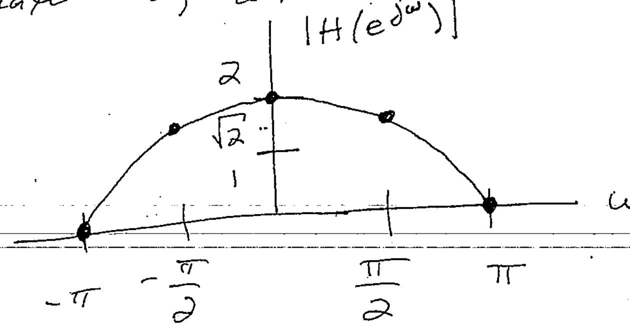
$$H(e^{j\omega}) = H[z]\Big|_{z=e^{j\omega}} = 1 + e^{-j\omega}$$

The magnitude of the frequency response (i.e., magnitude response) is

$$|H(e^{j\omega})| = |1 + e^{-j\omega}| = |-1 - e^{-j\omega}|$$

This represents the distance from the zero located at $z=-1$ and a point on the unit circle at $z = e^{-j\omega}$. So, we'll pick three frequencies to evaluate: 0, $\frac{\pi}{2}$, and π.

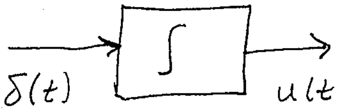




Lowpass filtering

2. Integrators in both time domains.

Continuous-time

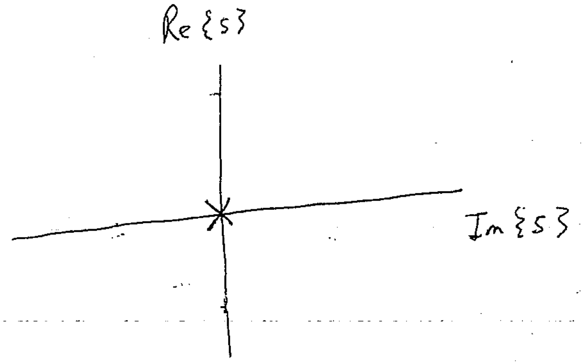


Normally, $y(t) = e^{\lambda t} u(t)$

Characteristic root of $\lambda = 0$.

Characteristic roots = poles.

Therefore, pole at $s = 0$.



$$H(s) = \frac{1}{s}$$

$$y(t) = \int_{0^-}^{t} x(\tau)\, d\tau$$

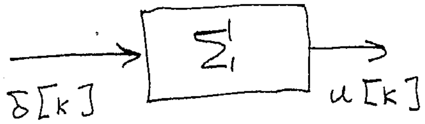
Take Laplace transforms of both sides with $y(0^-) = 0$:

$$Y(s) = \frac{1}{s} X(s) \quad \text{for } \mathrm{Re}\{s\} > 0$$

$$\frac{Y(s)}{X(s)} = \frac{1}{s} = H(s)$$

for $\mathrm{Re}\{s\} > 0$

Discrete-time



$y[k] = y[k-1] + x[k]$ with $y[-1] = 0$.

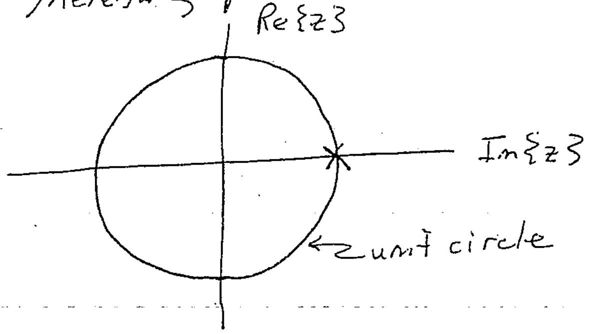
Let $x[k] = \delta[k]$, $y[k] = u[k]$.

$y[k] = \gamma^k u[k]$

Characteristic root of $\gamma = 1$.

Characteristic roots = poles.

Therefore, pole at $z = 1$.



$$H[z] = \frac{1}{1 - z^{-1}}$$

$$y[k] = y[k-1] + x[k]$$

Take z-transforms of both sides with $y[-1] = 0$:

$$Y[z] = z^{-1} Y[z] + X[z]$$

$$(1 - z^{-1})\, Y[z] = X[z]$$

$$\frac{Y[z]}{X[z]} = \frac{1}{1 - z^{-1}} = H[z]$$

for $|z| > 1$.

L-2

3. Oscillator in both time domains

Laplace Domain

$s = \sigma + j2\pi f$

Units of $2\pi f$ are rad/s

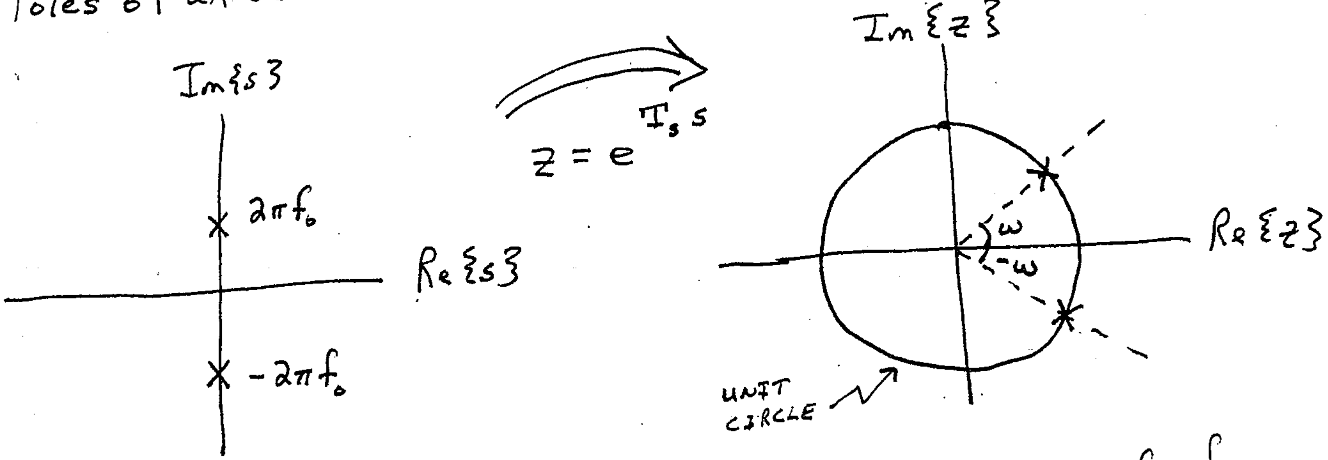
Poles of an oscillator

z-domain

$z = re^{j\omega}$

Units of ω are rad/sample

Poles of an oscillator?



$$H(s) = \frac{A}{s - j2\pi f_0} + \frac{A^*}{s + j2\pi f_0}$$

$$h(t) = Ae^{+j2\pi f_0 t} + \underbrace{A^* e^{-j2\pi f_0 t}}_{\text{conjugate of } Ae^{+j2\pi f_0}}$$

Let $A = |A|e^{j\theta}$,

$$h(t) = |A|e^{+j2\pi f_0 t + j\theta} + |A|e^{-j2\pi f_0 t - j\theta}$$

Using $e^{+j\Omega} + e^{-j\Omega} = 2\cos\Omega$,

$$h(t) = 2|A|\cos(2\pi f_0 t + \theta)$$

With $s = +j2\pi f_0$ for one pole, $s = -j2\pi f_0$ for the other pole, the poles in the z-domain become

$$z = e^{T_s(+j2\pi f_0)}$$ and

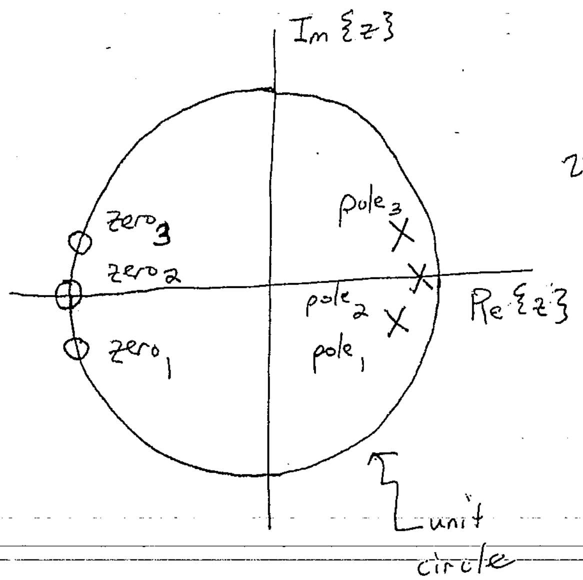
$$z = e^{T_s(-j2\pi f_0)}$$

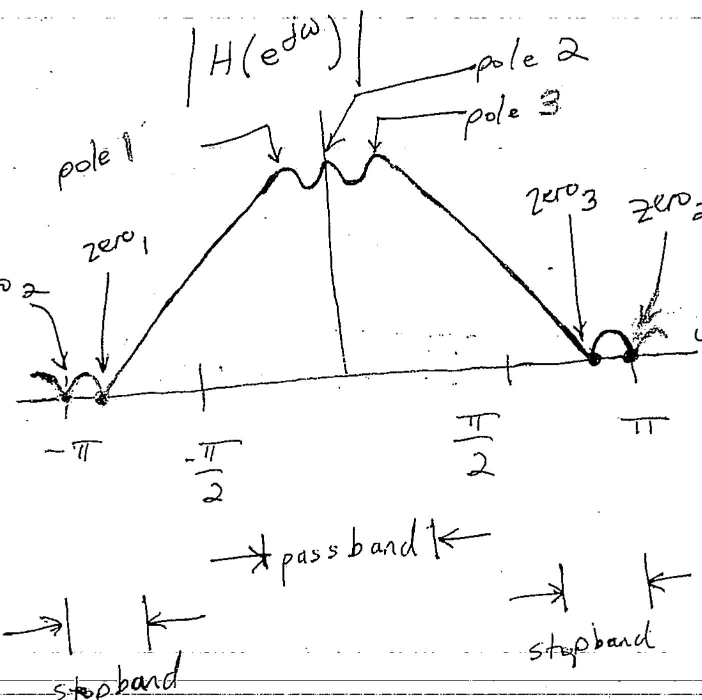
With $T_s = \frac{1}{f_s}$, the oscillator frequency is

$$\omega = 2\pi f_0 T_s = 2\pi \frac{f_0}{f_s}$$

which is in units of rad/sample.

4. Design a lowpass filter with poles and zeros by placing poles and zeros in the right locations. Filter order = 3. So, there are three zeros and three poles. We want real-valued coefficients in the numerator and denominator polynomials. So, with three poles, we'll have two conjugate symmetric poles, and one real-valued pole. Same goes for the zeros. Zeros on the unit circle indicate the stopband (frequency band that is strongly attenuated). Poles near the unit circle (but not on it) indicate the passband (frequency band that is passed without much if any attenuation).





The part not in the passband or stopband is a "don't care" region called the transition band.

5. Shifting in time property of the Laplace Transform

$$\mathcal{L}\{ f(t-t_0)\, u(t-t_0) \} = e^{-s t_0} F(s)$$

where $F(s) = \mathcal{L}\{ f(t)\, u(t) \}$

Example: $g(t) = e^{-4t} u(t-5)$

$$= e^{-20} e^{-4(t-5)} u(t-5)$$

$$= e^{-20} f(t-t_0)\, u(t-t_0)$$

with $t_0 = 5$, $G(s) = e^{-20} \dfrac{e^{-5s}}{s+4}$

Derivation of property:

$h(t) = f(t-t_0)\, u(t-t_0)$

$$H(s) = \int_{-\infty}^{\infty} f(t-t_0)\, u(t-t_0)\, e^{-st}\, dt$$

Let $v = t - t_0$, so $t = v + t_0$ and $dt = dv$,

$$H(s) = \int_{-\infty}^{\infty} f(v)\, u(v)\, e^{-s(v+t_0)}\, dv$$

$$= e^{-st_0} \int_{-\infty}^{\infty} f(v)\, u(v)\, e^{-sv}\, dv$$

$$= e^{-st_0} F(s)$$

where $F(s) = \mathcal{L}\{ f(t) u(t) \}$.

L-6

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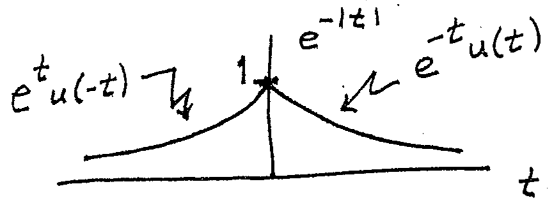
Second Review for Midterm #2 Prof. Brian L. Evans

1. Lathi, 4.8-2(a). Find the bilateral Laplace transform and corresponding region of convergence of $e^{-|t|}$:



$$e^{-|t|} = \begin{cases} e^{-t}u(t) & \text{for } t \geq 0 \\ e^{t}u(-t) & \text{for } t \leq 0 \end{cases}$$

$$f(t) = e^{-|t|} = e^{t}u(-t) + e^{-t}u(t) \quad \text{if} \quad u(t) = \begin{cases} 1 & \text{for } t > 0 \\ \frac{1}{2} & \text{if } t = 0 \\ 0 & \text{if } t < 0 \end{cases}$$

We need this definition of $u(t)$ in order for $f(0) = e^{-|0|} = 1 = e^{0}u(-0) + e^{-0}u(0) = 2\,u(0)$.

Bilateral Laplace Transform of $f(t)$:

$$F(s) = \mathcal{L}\{f(t)\} = \int_{-\infty}^{\infty} f(t)\,e^{-st}\,dt$$

where s is a complex variable and $F(s)$ is a complex-valued function of the complex-valued variable s.

Laplace transform is a linear operator. $\mathcal{L}\{0\} = 0$.

$$F(s) = \mathcal{L}\{e^{t}u(-t) + e^{-t}u(t)\}$$
$$= \mathcal{L}\{e^{t}u(-t)\} + \mathcal{L}\{e^{-t}u(t)\}$$

We know $\mathcal{L}\{e^{-t}u(t)\} = \frac{1}{s+1}$ for $Re\{s\} > -1$.

We would like to Know $\mathcal{L}\{e^{t}u(-t)\}$.

Let's derive the general property for reversing in time:

$$\mathcal{L}\{g(-t)\} = \int_{-\infty}^{\infty} g(-t)\, e^{-st}\, dt$$

Let $v = -t$, so $t = -v$; $dt = -dv$; $t \to -\infty \Rightarrow v \to \infty$; and
$t \to \infty \Rightarrow v \to -\infty$:

$$\mathcal{L}\{g(-t)\} = -\int_{\infty}^{-\infty} g(v)\, e^{+sv}\, dv$$

Recalling that $-\int_{a}^{-a} h(\lambda)\, d\lambda = \int_{-a}^{a} h(\lambda)\, d\lambda$,

$$\mathcal{L}\{g(-t)\} = \int_{-\infty}^{\infty} g(v)\, e^{-(-s)v}\, dv = G(-s)$$

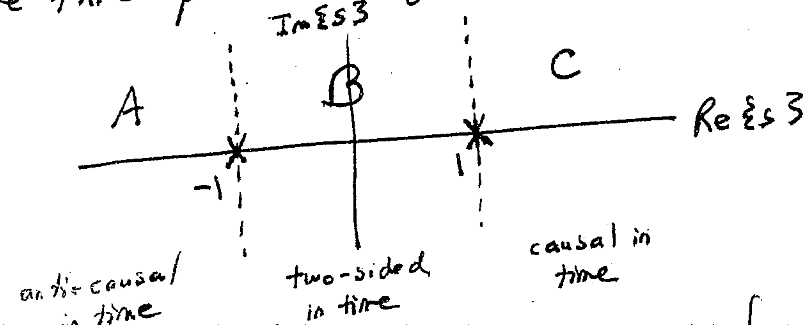
Note that the region of convergence is reversed as well.

Back to the original problem,

$$F(s) = \mathcal{L}\{e^{t}u(-t)\} + \mathcal{L}\{e^{-t}u(t)\}$$

$$= \underbrace{\frac{1}{-s+1}}_{Re\{s\}<1} + \underbrace{\frac{1}{s+1}}_{Re\{s\}>-1} = \underbrace{\frac{-1}{s-1} + \frac{1}{s+1}}_{-1<Re\{s\}<1}$$

There are three possible regions of convergence: A, B, C



Region of convergence information is necessary for the bilateral Laplace transform to guarantee a unique inverse. See Lathi, example 4.24, on page 334.

2. What is the value of $u(t)$ at $t=0$?

$$u(t) = \int_{-\infty}^{t} \delta(\lambda)\, d\lambda$$

$$\delta(t) \longrightarrow \boxed{\int} \longrightarrow u(t)$$

$$u(0) = \int_{-\infty}^{0} \delta(\lambda)\, d\lambda$$ is ambiguous.

If $t>0$, let $v>0$,

$$u(t) = \int_{-\infty}^{0+v} \delta(\lambda)\, d\lambda = 1$$

If $t<0$, let $v<0$,

$$u(t) = \int_{-\infty}^{0-v} \delta(\lambda)\, d\lambda = 0$$

3. Laplace transform of a signal that is flipped and shifted in time?

$$\mathcal{L}\{ g(-t+t_0)\} = \mathcal{L}\{ g(-(t-t_0))\}$$

Apply Laplace transform properties twice:

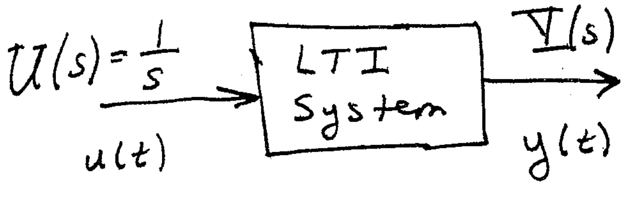
$$\mathcal{L}\{ g(-(t-t_0))\} = \mathcal{L}\{ g(t-t_0)\}\Big|_{s=-s}$$

$$= \left[e^{-st_0}\, G(s)\right]_{s=-s}$$

$$= e^{st_0}\, G(-s)$$

4. Midterm #2, Fall 1999, Problem 2.2, Variation

We are given that the step response of an LTI system is $4(1-e^{-5t})u(t) = 4u(t) - 4e^{-5t}u(t)$



$$Y(s) = \frac{4}{s} - \frac{4}{s+5} = \frac{20}{s(s+5)}$$

$$y(t) = 4u(t) - 4e^{-5t}u(t)$$

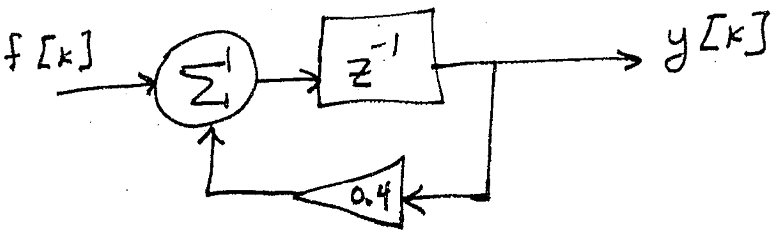
Find the impulse response $h(t)$ of the LTI system.

The transfer function, $H(s)$, is the Laplace transform of the impulse response; hence, $h(t) = \mathcal{L}^{-1}\{H(s)\}$.

$$H(s) = \frac{Y(s)}{U(s)} = \frac{\frac{20}{s(s+5)}}{\frac{1}{s}} = \frac{20}{s+5}$$

$$h(t) = \mathcal{L}^{-1}\left\{\frac{20}{s+5}\right\} = 20e^{-5t}u(t)$$

5. Lathi, 5.5-1(a). Find the amplitude (magnitude) and the phase of frequency response of the following LTI system:



Here, the block $\boxed{z^{-1}}$ represents a delay by one sample. Recall in solving difference equations, one took the z-transform of both sides (assuming zero initial conditions):

$$y[k] - 0.4\,y[k-1] = f[k-1]$$

$$Y[z] - 0.4\,\underbrace{z^{-1}}_{\text{delay by one sample}} Y[z] = \underbrace{z^{-1}}_{} F[z]$$

The frequency response can be determined from the transfer function as follows:

$$H[e^{j\omega}] = H[z]\Big|_{z=e^{j\omega}}$$

Here, ω is in units of rad/sample. The difference equation governing the block diagram and its z-transform:

$$y[k] = \left(0.4\,y[k] + f[k]\right)_{k \to k-1}$$

$$y[1 = 0.4\,y[k-1] + f[k-1]$$

$$Y[z] = 0.4\,z^{-1}Y[z] + z^{-1}F[z]$$

Now, we need to isolate $Y[z]$ on one side, and $F[z]$ on the other:

$$(1 - 0.4z^{-1})\,Y[z] = z^{-1}F[z]$$

$$H[z] = \frac{Y[z]}{F[z]} = \frac{z^{-1}}{1 - 0.4z^{-1}} = \frac{1}{z - 0.4}$$

The frequency response is

$$H[e^{j\omega}] = H[z]\Big|_{z=e^{j\omega}} = \frac{1}{e^{j\omega} - 0.4}$$

The magnitude response is

$$|H[e^{j\omega}]| = \left|\frac{1}{\cos\omega + j\sin\omega - 0.4}\right| = \frac{1}{\sqrt{(0.4+\cos\omega)^2 + \sin^2\omega}}$$

Recall the following properties for absolute value:

$V_1 = r_1 e^{j\theta_1}$ $\qquad |V_1 V_2| = |V_1| \cdot |V_2| = r_1 r_2$

$V_2 = r_2 e^{j\theta_2}$ $\qquad \left|\frac{V_1}{V_2}\right| = \frac{|V_1|}{|V_2|} = \frac{r_1}{r_2}$

assuming $r_1 > 0$
and $r_2 > 0$

Recall the following properties for phase of complex numbers:

$\measuredangle V_1 = \theta_1$

$\measuredangle V_2 = \theta_2$

$$\measuredangle V_1 V_2 = \measuredangle r_1 r_2 e^{j(\theta_1 + \theta_2)}$$
$$= \theta_1 + \theta_2$$
$$\measuredangle \frac{V_1}{V_2} = \measuredangle \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)} = \theta_1 - \theta_2$$

So, the phase response is

$$\measuredangle H[e^{j\omega}] = \measuredangle 1 - \measuredangle (e^{j\omega} - 0.4)$$
$$= \measuredangle (\cos\omega + j\sin\omega - 0.4)$$
$$= \arctan\left(\frac{\sin\omega}{\cos\omega - 0.4}\right)$$